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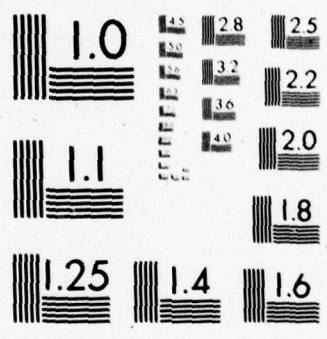
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SCHEME-INDEPENDENT STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS
OF HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS. I

by

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ABSTRACT. Easily checkable sufficient stability criteria are obtained for explicit dissipative approximations to mixed initial-boundary value problems associated with the system $u_t = Au_x$ in the quarter plane $x \geq 0, t \geq 0$. The criteria are given entirely in terms of the boundary conditions for the outflow unknowns. The results imply that certain well known boundary conditions, when used in combination with any (stable) dissipative scheme, always maintain stability.

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0. Introduction

This paper provides a stability study of a wide class of difference approximations for a hyperbolic mixed initial-boundary value problem in the quarter plane $x \geq 0, t \geq 0$. The approximated differential system is $u_t = Au_x$ where A is diagonal, and the inflow and outflow unknowns interact at the boundary. For the difference approximation we consider general dissipative explicit two-level schemes, with general boundary conditions which determine the boundary values in terms of outflow values at interior points. We show that such boundary conditions may have arbitrary degree of accuracy. This is included in Section 1.

In Section 2 we begin to discuss stability. We show that the entire approximation is stable if and only if the scalar components of its outflow part are; thus reducing the stability question to that of a scalar outflow problem. From that point on, our aim will be to obtain easily checkable sufficient stability criteria for the reduced problem. All our results are scheme-independent and are given exclusively in terms of the outflow boundary conditions. The only such result that we know of, is due to Kreiss ([3]; see also [1]) who proved that for dissipative schemes, boundary extrapolation always maintains stability.

In the remainder of Section 2 we state our main result and consider several examples. The main result is for the case where the outflow boundary conditions are translatory, i.e., determined at all boundary points by the same procedure. The result states that if the outflow boundary conditions are generated by a solvable stable scheme, then the entire approximation is stable, independently of the interior scheme. The examples considered show that if the outflow boundary conditions are generated by oblique extrapolation, by explicit or implicit Euler schemes,

or by the Box-Scheme, then overall stability is assured.

In Section 3 we derive a preliminary scheme-independent result, which is an explicit interpretation of Kreiss' determinantal stability criteria [4]. This result, which seems to be of independent interest, is used in Section 4 to derive our stability criteria for the case of translatory boundary conditions.

The theoretical basis for our work was given by Kreiss [4] and by Gustafsson, Kreiss and Sundström [2]. We assume that the reader is familiar with these papers.

In a forthcoming paper we discuss the extension of our results to nondissipative and multi-level schemes.

We are grateful to Bjorn Engquist for many helpful discussions.

1. The Difference Approximation

Consider a first order hyperbolic system of partial differential equations

$$(1.1a) \quad \partial u / \partial t = A \partial u / \partial x, \quad x \geq 0, t \geq 0,$$

where $u = (u^{(1)}(x,t), \dots, u^{(n)}(x,t))'$ is the transposed vector of unknowns, and A is a constant $n \times n$ Hermitian matrix of the form

$$A = \begin{pmatrix} A^I & 0 \\ 0 & A^{II} \end{pmatrix}, \quad A^I < 0, A^{II} > 0.$$

Without restriction we may assume that A is diagonal.

The solution of (1.1a) is uniquely determined if initial values

$$(1.1b) \quad u(x,0) = f(x), \quad x \geq 0,$$

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and boundary conditions

$$(1.1c) \quad u^I(0, t) = Su^{II}(0, t) + g(t), \quad t \geq 0,$$

are prescribed. Here the partition

$$u^I = (u^{(1)}, \dots, u^{(l)})', \quad u^{II} = (u^{(l+1)}, \dots, u^{(n)})'$$

corresponds to that of A , and S is an $l \times (n-l)$ rectangular matrix.

To solve the initial-boundary value problem (1) by a difference approximation we introduce a mesh-size $h = \Delta x > 0$, $k = \Delta t > 0$, such that $\lambda = k/h = \text{constant}$. Using the standard notation $v_v(t) = v(vh, t)$, we approximate (1.1a) by a consistent two-sided difference scheme of the form

$$(1.2a) \quad v_v(t+k) = Qv_v(t), \quad v = r, r+1, \dots,$$

$$Q = \sum_{j=-r}^p A_j E^j, \quad Ev_v = v_{v+1},$$

with initial values

$$(1.2b) \quad v_v(0) = f_v, \quad v = 0, 1, 2, \dots$$

The A_j are fixed diagonal $n \times n$ matrices depending on A and on λ such that A_{-r}, A_p are nonsingular.

Throughout the paper we assume that scheme (1.2a) is dissipative, i.e., for some $\delta > 0$ and positive integer ω , the eigenvalues $\theta(\xi)$ of the amplification matrix

$$\hat{Q}(\xi) = \sum_{j=-r}^p A_j e^{ij\xi}, \quad -\pi < \xi \leq \pi,$$

satisfy

$$|\theta(\xi)| \leq 1 - \delta |\xi|^{2\omega}.$$

Since the A_j are diagonal, dissipativity guarantees that the scheme (1.2a) is stable.

In order to uniquely determine the solution of (1.2), we must specify, at each time step, the r boundary values $v_\mu(t)$, $\mu = 0, 1, \dots, r-1$. For the outflow unknowns we do it by boundary conditions of the form

$$(1.3) \quad \sum_{j=0}^s c_{\mu j}^{(1)} v_{\mu+j}^{II}(t+k) = \sum_{j=0}^s c_{\mu j}^{(0)} v_{\mu+j}^{II}(t), \quad \mu = 0, \dots, r-1,$$

$c_{\mu j}$ being fixed diagonal $(n-l) \times (n-l)$ matrices. For the inflow part we use the physical boundary condition

$$(1.4a) \quad v_0^I(t) = S v_0^{II}(t) + g(t)$$

together with $r-1$ additional conditions of the form

$$(1.4b) \quad v_\mu^I(t) = \sum_{j=0}^q D_{\mu j} v_j^{II}(t) + g_\mu(t), \quad \mu = 1, \dots, r-1,$$

where $D_{\mu j}$ are fixed $l \times (n-l)$ matrices and the $g_\mu(t)$ depend on h and on $g(t)$.

It is well known that using conditions of the general form (1.3), one can achieve at the boundary arbitrary degrees of accuracy. We note that this is true also for conditions of type (1.4b). In fact, if accuracy of order d is desired, then using the differential system and (1.1c), we find that a Taylor expansion of a smooth solution of (1.1) yields

$$\begin{aligned}
u_{\mu}^I(t) &= \sum_{j=0}^d \frac{(\mu h)^j}{j!} \frac{\partial^j}{\partial x^j} u^I(0, t) + o(h^{d+1}) \\
&= \sum_{j=0}^d \frac{(\mu h)^j}{j!} (A^I)^{-j} \frac{\partial^j}{\partial t^j} u^I(0, t) + o(h^{d+1}) \\
&= \sum_{j=0}^d \frac{(\mu h)^j}{j!} (A^I)^{-j} \left[S \frac{\partial^j}{\partial t^j} u^{II}(0, t) + \frac{d^j}{dt^j} g(t) \right] + o(h^{d+1}) \\
&= \sum_{j=0}^d \frac{(\mu h)^j}{j!} (A^I)^{-j} \left[S(A^{II})^j \frac{\partial^j}{\partial x^j} u^{II}(0, t) + \frac{d^j}{dt^j} g(t) \right] + o(h^{d+1}).
\end{aligned}$$

Thus, (1.4b) follows upon approximating $\partial^j / \partial x^j u^{II}(0, t)$ by linear combinations of $u_0^{II}(t), \dots, u_q^{II}(t)$ of the right accuracy.

For example, if $v_1^I(t)$ is required to second order of accuracy, we may use

$$\frac{\partial u^{II}(0, t)}{\partial x} \sim \frac{-3u_0^{II}(t) + 4u_1^{II}(t) - u_2^{II}(t)}{2h}, \quad \frac{\partial^2 u^{II}(0, t)}{\partial x^2} \sim \frac{u_0^{II}(t) - 2u_1^{II}(t) + u_2^{II}(t)}{2h^2},$$

to obtain a second order accurate boundary condition

$$v_1^I(t) = D_0 v_0^{II}(t) + D_1 v_1^{II}(t) + D_2 v_2^{II}(t) + g_1(t),$$

where

$$D_0 = M_0 - \frac{3}{2} M_1 + \frac{1}{4} M_2, \quad D_1 = 2M_1 - \frac{1}{2} M_2, \quad D_2 = -\frac{1}{2} M_1 + \frac{1}{4} M_2,$$

$$M_j = (A^I)^{-j} S(A^{II})^j, \quad j = 0, 1, 2,$$

$$g_1(t) = g(t) + h(A^I)^{-1} \dot{g}(t) + \frac{h^2}{2} (A^I)^{-2} \ddot{g}(t).$$

2. Statement of Main Result and Examples

The difference approximation is completely defined now by the dissipative (stable) scheme (1.2) together with the boundary conditions (1.3), (1.4), and we raise the question of overall stability in the sense of Gustafsson et al. ([2], Definition 3.3).

Since the A_j are diagonal, we can split the scheme (1.2a) into its inflow and outflow parts,

$$(2.1) \quad v_v^I(t+k) = \sum_{j=-r}^p A_j^I v_{v+j}^I(t), \quad v = r, r+1, \dots,$$

and

$$(2.2) \quad v_v^{II}(t+k) = \sum_{j=-r}^p A_j^{II} v_{v+j}^{II}(t), \quad v = r, r+1, \dots,$$

where

$$A_j = \begin{pmatrix} A_j^I & 0 \\ 0 & A_j^{II} \end{pmatrix}, \quad -r \leq j \leq p,$$

corresponds to the partition of A . We immediately see that the outflow problem (2.2), (1.3) is self contained, while the inflow problem (2.1), (1.4) depends on the outflow part only to the extent that the outflow computations provide the inhomogeneous boundary values in (1.4). Thus, stability of the entire approximation is equivalent to the following separate questions:

- (a) Stability of the inflow problem (2.1) with inhomogeneous boundary values.
- (b) Stability of the outflow problem (2.2) (1.3).

Since the stability definition (3.3) of [2] gives bounds for the inhomogeneous boundary values, it suffices, for the outflow problem, to consider homogeneous boundary values. The A_j^I are diagonal, hence the problem splits into ℓ independent dissipative approximations with homogeneous boundary values. It was shown by Kreiss ([3], Theorem 5) that such scalar approximations are stable independently of the basic scheme. Thus the inflow problem is stable and it remains to consider (b). Since the A_j^{II} of (2.2) and the $C_{\mu j}$ of (1.3) are diagonal, the outflow problem splits as well.

We conclude that approximation (1.2) - (1.4) is stable if and only if the $n-l$ scalar components of its outflow part are stable. So, from now on, we may restrict the stability discussion to the following scalar case where we approximate an initial-value problem

$$(2.3) \quad \frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x}, \quad a > 0, \quad x \geq 0, \quad t \geq 0, \quad u(x, 0) = f(x),$$

by a consistent dissipative scheme

$$(2.4a) \quad \begin{aligned} v_v(t+k) &= Qv_v(t), & v &= r, r+1, \dots, \\ Q &= \sum_{j=-r}^R a_j E^j, & E v_v &= v_{v+1}, \end{aligned}$$

together with initial values

$$(2.4b) \quad v_v(0) = f_v, \quad v = 0, 1, 2, \dots,$$

and boundary conditions

$$(2.5) \quad \sum_{j=0}^s c_{\mu j}^{(1)} v_{\mu+j}(t+k) = \sum_{j=0}^s c_{\mu j}^{(0)} v_{\mu+j}(t), \quad \mu = 0, \dots, r-1.$$

Here the a_j and the $c_{\mu j}$ are constants depending on a and on λ .

Our purpose is to provide easily checkable sufficient stability criteria for the approximation (2.4)(2.5) which depend entirely on the boundary conditions (2.5). Particularly effective criteria of that nature are obtained when the boundary conditions are translatory, i.e., of the form

$$(2.6) \quad \sum_{j=0}^s c_j^{(1)} v_{\mu+j}(t+\Delta t) = \sum_{j=0}^s c_j^{(0)} v_{\mu+j}(t), \quad \mu = 0, \dots, r-1,$$

where the c_j are independent of μ . In order to state our main result we introduce the boundary-scheme

$$\begin{aligned}
 (2.7) \quad T^{(1)} v_v(t+k) &= T^{(0)} v_v(t), & v &= 0, \pm 1, \pm 2, \dots, \\
 T^{(\alpha)} &= T^{(\alpha)}(E) = \sum_{j=0}^s c_j^{(\alpha)} E^j, & \alpha &= 0, 1, \quad E v_v = v_{v+1}
 \end{aligned}$$

which generates the boundary conditions (2.6) upon restriction of v to the values $0, 1, \dots, r-1$.

We call the boundary scheme stable if it is stable when applied to all grid points vh , $-\infty < v < \infty$. This leads to

THEOREM 2.1 (The Main Theorem). Approximation (2.4)(2.6) is stable if the boundary-scheme (2.7) is stable and if

$$(2.8) \quad T^{(1)}(\kappa) = \sum_{j=0}^s c_j^{(1)} \kappa^j \neq 0 \quad \forall |\kappa| \leq 1.$$

The stability criterion in this theorem is independent of the basic scheme. The proof, as well as other scheme-independent results for the translatory case, are given in Section 4.

Often, the boundary-scheme is known in advance to be stable. Thus, in applying Theorem 2.1, it only remains to verify the solvability-condition (2.8).

If the boundary conditions are explicit, i.e., of the form

$$(2.9) \quad v_\mu(t+k) = \sum_{j=0}^s c_j v_{\mu+j}(t), \quad \mu = 0, \dots, r-1,$$

then $T^{(1)}(\kappa) \equiv 1$. Hence (2.8) is automatically fulfilled and Theorem 2.1 reduces to

COROLLARY 2.2. If the boundary conditions (2.9) are generated by a stable scheme, then approximation (2.4)(2.9) is stable.

EXAMPLE 1. Determine the boundary conditions by oblique extrapolation:

$$(2.10) \quad v_{\mu}(t+k) = v_{\mu+1}(t), \quad \mu = 0, \dots, r-1.$$

Clearly, the generating boundary-scheme is explicit and unconditionally stable; so by Corollary 2.2 the stability of (2.4)(2.10) is assured.

Note that (2.10) is inconsistent with the differential equation (2.3) unless $\lambda a = 1$ in which case (2.10) coincides with our next example.

EXAMPLE 2. Let the boundary conditions be generated by the right-sided explicit Euler scheme, i.e.,

$$(2.11) \quad v_{\mu}(t+k) = v_{\mu}(t) + \lambda a[v_{\mu+1}(t) - v_{\mu}(t)], \quad \mu = 0, \dots, r-1.$$

Since the basic scheme (2.4a) must satisfy the Courant-Friedrichs-Levi condition $\lambda a \leq 1$, Euler's scheme is stable too, and by Corollary 2.2 the stability of (2.4)(2.11) follows.

EXAMPLE 3. Take

$$(2.12) \quad v_{\mu}(t+k) - \lambda a[v_{\mu+1}(t+k) - v_{\mu}(t+k)] = v_{\mu}(t),$$

$$\mu = 0, \dots, r-1,$$

which is generated by the right-sided unconditionally stable implicit Euler scheme. To comply with Theorem 2.1, we must have

$$T^{(1)}(\kappa) \equiv 1 - \lambda a(\kappa - 1) \neq 0 \quad \forall |\kappa| \leq 1.$$

Since $\lambda a > 0$, then κ with $|\kappa| \leq 1$ gives

$$\operatorname{Re} T^{(1)}(\kappa) = 1 + \lambda a(1 - \operatorname{Re} \kappa) \geq 1,$$

so (2.5)(2.12) is stable.

EXAMPLE 4. We use the unconditionally stable Box-Scheme to generate

$$\begin{aligned}
 (2.13) \quad v_{\mu}(t+k) + v_{\mu+1}(t+k) - \lambda a[v_{\mu+1}(t+k) - v_{\mu}(t+k)] \\
 = v_{\mu}(t) + v_{\mu+1}(t) + \lambda a[v_{\mu+1}(t) - v_{\mu}(t)], \quad \mu = 0, \dots, r-1.
 \end{aligned}$$

Since

$$\operatorname{Re} T^{(1)}(\kappa) = 1 + \operatorname{Re} \kappa + \lambda a(1 - \operatorname{Re} \kappa) > 0 \quad \forall |\kappa| \leq 1,$$

then by Theorem 2.1 again, (2.4)(2.13) is stable.

In view of the previous stability discussion, Examples 1-4 imply that if the boundary conditions (1.3) are generated by oblique extrapolation, explicit or implicit right-sided Euler schemes, or by the Box-Scheme, then the entire approximation (1.2) - (1.4) is stable.

The boundary conditions in Examples 1-4 were studied by Gustafsson et al. [2] and by Skölleremo [5] in combination with specific 3-point basic schemes.

3. A Preliminary Determinantal Criteria

In order to investigate the stability of (2.4), (2.5) we introduce the space $l_2(h)$ of all grid functions

$$w = \{w_v\}_{v=0}^{\infty} \quad \text{with} \quad \|w\|^2 \equiv h \sum_{v=0}^{\infty} |w_v|^2 < \infty.$$

We write the approximation in operator form

$$v(t+k) = Gv(t),$$

where $G: l_2(h) \rightarrow l_2(h)$ is uniquely determined by (2.4a) and (2.5).

Following Kreiss ([4], Main Theorem) we will show that G has no eigenvalues z with $|z| \geq 1$. This will suffice to assure stability both in the sense of [4] and in the sense of Definition 3.3 of [2].

To check the eigenvalues of G we must adopt Kreiss' recipe in [4]: If z with $|z| \geq 1$ is an eigenvalue, then for some nontrivial $g \in \ell_2(h)$, $Gg = zg$. Thus, by (2.4a) and (2.5), g must satisfy the resolvent equation

$$(3.1) \quad zg_v = \sum_{j=-r}^p a_j g_{v+j}, \quad v = r, r+1, \dots,$$

and the boundary relations

$$(3.2) \quad z \sum_{j=0}^s c_{\mu j}^{(1)} g_{\mu+j} = \sum_{j=0}^s c_{\mu j}^{(0)} g_{\mu+j}, \quad \mu = 0, \dots, r-1.$$

The most general solution of (3.1) in $\ell_2(h)$ can be written as

$$(3.3) \quad g_v = \sum_{\alpha=1}^k \sum_{\beta=0}^{m_\alpha-1} \sigma_{\alpha\beta} P_{\alpha\beta}(v) \kappa_\alpha^\beta, \quad v \geq 0.$$

Here the $\kappa_\alpha = \kappa_\alpha(z)$, $1 \leq \alpha \leq k$, are the distinct roots of the characteristic equation

$$(3.4) \quad \sum_{j=-r}^p a_j \kappa^j - z = 0,$$

which satisfy $0 < |\kappa_\alpha| < 1$, each with multiplicity $m_\alpha = m_\alpha(z)$; the $P_{\alpha\beta}(v)$ are arbitrary polynomials in v with $\deg[P_{\alpha\beta}(v)] = \beta$; and the $\sigma_{\alpha\beta}$ are free coefficients yet to be determined. Since (2.4a) is dissipative and consistent with (2.3), we use Lemma 2 and part of the proof of Lemma 7 of [4] to find that for z with $|z| \geq 1$, equation (3.4) has precisely r roots κ with $0 < |\kappa| < 1$. Thus g_v of (3.3) depends on r parameters. Substituting (3.3) in (3.2), we obtain a linear homogeneous system of r equations with the r unknowns $\sigma_{\alpha\beta}$. The system is of the form $J\sigma' = 0$ where $J = J(z)$ is an $r \times r$ matrix and σ is the unknown vector. This yields

LEMMA 3.1 (Lemma 3, [4]). z with $|z| \geq 1$ is an eigenvalue of G if and only if $\det J(z) \neq 0$.

Going through the above process, we make a particular choice of the polynomials $P_{\alpha\beta}(\nu)$ that leads to an explicit expression of J which later proves useful. We choose

$$(3.5) \quad P_{\alpha\beta}(\nu) = \kappa_{\alpha}^{-\beta} \beta! (\nu)_{\beta}.$$

Inserting (3.5) in (3.3) and then in (3.2) we obtain

$$(3.6) \quad \sum_{\alpha=1}^k \sum_{\beta=0}^{m_{\alpha}-1} \sum_{j=0}^s [z c_{\mu j}^{(1)} - c_{\mu j}^{(0)}] \beta! (\nu)_{\beta} \kappa_{\alpha}^{\mu+j-\beta} \sigma_{\alpha\beta} = 0, \\ \mu = 0, \dots, r-1,$$

which constitutes the system $J\sigma' = 0$.

At this point we associate with the boundary conditions (2.5) a set of polynomial boundary-functions

$$(3.7) \quad b_{\mu}(z, \kappa) = \sum_{j=0}^s [z c_{\mu j}^{(1)} - c_{\mu j}^{(0)}] \kappa^{\mu+j}, \quad \mu = 0, \dots, r-1.$$

Since

$$\frac{\partial^{\beta} b_{\mu}(z, \kappa)}{\partial \kappa^{\beta}} = \sum_{j=0}^s [z c_{\mu j}^{(1)} - c_{\mu j}^{(0)}] \beta! (\nu)_{\beta} \kappa^{\mu+j-\beta},$$

then system (3.6) may be written as

$$(3.8) \quad \sum_{\alpha=1}^k \sum_{\beta=0}^{m_{\alpha}-1} \frac{\partial^{\beta} b_{\mu}(z, \kappa_{\alpha})}{\partial \kappa_{\alpha}^{\beta}} \sigma_{\alpha\beta} = 0, \quad \mu = 0, \dots, r-1.$$

Thus the coefficient matrix J takes the form

$$(3.9a) \quad J = [B(z, \kappa_1, m_1), \dots, B(z, \kappa_k, m_k)]$$

where the $B(z, \kappa_{\alpha}, m_{\alpha})$, $1 \leq \alpha \leq k$, are $r \times m_{\alpha}$ blocks given by

(3.9b)

$$B(z, \kappa_\alpha, m_\alpha) = \left[\begin{bmatrix} b_0(z, \kappa) \\ b_1(z, \kappa) \\ \vdots \\ b_{r-1}(z, \kappa) \end{bmatrix}, \frac{\partial}{\partial \kappa} \begin{bmatrix} b_0(z, \kappa) \\ b_1(z, \kappa) \\ \vdots \\ b_{r-1}(z, \kappa) \end{bmatrix}, \dots, \frac{\partial^{m_\alpha-1}}{\partial \kappa^{m_\alpha-1}} \begin{bmatrix} b_0(z, \kappa) \\ b_1(z, \kappa) \\ \vdots \\ b_{r-1}(z, \kappa) \end{bmatrix} \right]_{\kappa=\kappa_\alpha}$$

Defining a partition of r to be any set of positive integers $\{m_i\}_{i=1}^N$ which satisfies $m_1 + \dots + m_N = r$, we state

THEOREM 3.2. Approximation (2.4)(2.5) is stable if for every z with $|z| \geq 1$, every partition $\{m_i\}_{i=1}^N$ of r , and every set of distinct values $\{\kappa_i\}_{i=1}^N$ with $0 < |\kappa_i| < 1$,

$$(3.10) \quad \det[B(z, \kappa_1, m_1), \dots, B(z, \kappa_N, m_N)] \neq 0.$$

Proof. Take an arbitrary z_0 with $|z_0| \geq 1$ and let $\kappa_\alpha(z_0)$, $1 \leq \alpha \leq k$, be the distinct roots of (3.1), each with multiplicity $m_\alpha(z_0)$. Since $\{m_\alpha(z_0)\}_{\alpha=1}^k$ is a partition of r , then (3.10) holds for our z_0 , $\kappa_\alpha(z_0)$ and $m_\alpha(z_0)$. Thus, by Lemma 3.1, z_0 is not an eigenvalue of G , and by Kreiss Main Theorem in [4], stability follows.

Since the determinant in (3.10) depends entirely on the boundary functions (3.7), Theorem 3.2 is scheme-independent. Thus in applying the theorem, one avoids the inherent difficulty of solving the characteristic equation (3.4).

4. Translatory Boundary Conditions

In this section we return to consider the approximation (2.4)(2.6) where the boundary conditions are translatory.

The boundary-functions associated with (2.6) are

$$b_{\mu}(z, \kappa) = \sum_{j=0}^s (zc_j^{(1)} - c_j^{(0)}) \kappa^{\mu+j}, \quad \mu = 0, \dots, r-1.$$

Thus writing $b(z, \kappa) = b_0(z, \kappa)$ we have

$$b_{\mu}(z, \kappa) = \kappa^{\mu} b(z, \kappa), \quad \mu = 0, \dots, r-1.$$

Consequently, if $\{m_i\}_{i=1}^N$ is a partition of r , then the $r \times r$ matrix

$$(4.1) \quad [B(z, \kappa_1, m_1), \dots, B(z, \kappa_N, m_N)]$$

in (3.10) is given by the $r \times m_i$ blocks

$$B(z, \kappa_i, m_i) = \left[\begin{array}{c} b(z, \kappa) \\ \kappa b(z, \kappa) \\ \vdots \\ \kappa^{r-1} b(z, \kappa) \end{array} \right], \frac{\partial}{\partial \kappa} \left[\begin{array}{c} b(z, \kappa) \\ \kappa b(z, \kappa) \\ \vdots \\ \kappa^{r-1} b(z, \kappa) \end{array} \right], \dots, \frac{\partial^{m_i-1}}{\partial \kappa^{m_i-1}} \left[\begin{array}{c} b(z, \kappa) \\ \kappa b(z, \kappa) \\ \vdots \\ \kappa^{r-1} b(z, \kappa) \end{array} \right] \Big|_{\kappa=\kappa_i}$$

The fact that (4.1) is determined now by the single boundary-function $b(z, \kappa)$, implies the following significant simplification of Theorem 3.2.

THEOREM 4.1. Approximation (2.4)(2.6) is stable if for every z with $|z| \geq 1$ and κ with $0 < |\kappa| < 1$, we have

$$(4.2) \quad b(z, \kappa) = \sum_{j=0}^s (zc_j^{(1)} - c_j^{(0)}) \kappa^j \neq 0.$$

Proof. Take an arbitrary z with $|z| \geq 1$, a partition $\{m_i\}_{i=1}^N$ of r , and distinct values κ_i , $1 \leq i \leq N$, with $0 < |\kappa_i| < 1$. In order to prove stability, it suffices, by Theorem 3.2, to verify (3.10).

For this purpose, let

$$(4.3) \quad \sum_{\mu=0}^{r-1} \gamma_{\mu} \left[\begin{array}{c} \kappa_1^{\mu} b(z, \kappa_1) \\ \vdots \\ \frac{\partial^{m_N-1}}{\partial \kappa_N^{m_N-1}} [\kappa_N^{\mu} b(z, \kappa_N)] \end{array} \right] = 0,$$

be a vanishing linear combination of the rows of (4.1). The vector relation in (4.3) consists of r scalar equations,

$$\left\{ \sum_{\mu=0}^{r-1} \gamma_{\mu} \frac{\partial^j}{\partial \kappa^j} [\kappa^{\mu} b(z, \kappa)] \right\}_{\kappa=\kappa_i} = 0, \quad 1 \leq i \leq N, \quad 0 \leq j \leq m_i - 1,$$

which we write as

$$(4.4) \quad \frac{\partial^j}{\partial \kappa^j} \left\{ b(z, \kappa) \left[\sum_{\mu=0}^{r-1} \gamma_{\mu} \kappa^{\mu} \right] \right\}_{\kappa=\kappa_i}, \quad 1 \leq i \leq N, \quad 0 \leq j \leq m_i - 1.$$

Since $0 < |\kappa_i| < 1$, then by hypothesis, the expression in the left brackets of (4.4) satisfies

$$[b(z, \kappa)]_{\kappa=\kappa_i} \neq 0, \quad 1 \leq i \leq N.$$

Thus, expanding (4.4) by Leibniz' rule and using induction on $j \geq 0$, we find that the sum in (4.4) have vanishing derivatives at $\kappa = \kappa_i$, i.e.,

$$\frac{d^j}{d\kappa^j} \left[\sum_{\mu=0}^{r-1} \gamma_{\mu} \kappa^{\mu} \right]_{\kappa=\kappa_i} = 0, \quad 1 \leq i \leq N, \quad 0 \leq j \leq m_i - 1.$$

We conclude that the polynomial

$$P(\kappa) \equiv \sum_{\mu=0}^{r-1} \gamma_{\mu} \kappa^{\mu},$$

which is of degree $r-1$ at most, has r roots; κ_i , $1 \leq i \leq N$, each with multiplicity m_i . Hence, $P(\kappa) \equiv 0$ and the coefficients γ_{μ} must vanish. By (4.3), therefore, the rows of (4.1) are linearly independent, so (3.10) holds, and stability follows.

Before turning to the proof of the Main Theorem (Theorem 2.1) we note that Theorem 4.1 applies also to single-leveled boundary conditions. For example, we can immediately obtain the following result of Kreiss [3] (see also [1]): Let the boundary conditions be determined by extrapolation of arbitrary degree $s-1$, i.e.,

$$(4.5) \quad \sum_{j=0}^s \binom{s}{j} (-1)^j v_{\mu+j}(t) = 0, \quad \mu = 0, \dots, r-1.$$

The associated boundary-function is

$$b(\kappa) = - \sum_{j=0}^s \binom{s}{j} (-1)^j \kappa^j = -(1-\kappa)^s.$$

Thus $b(\kappa) \neq 0$ for $0 < |\kappa| < 1$, and by Theorem 4.1, (2.4)(4.5) is stable.

Proof of Theorem 2.1. The amplification factor $\hat{T}(\xi)$ of the boundary-scheme (2.7) is given by

$$\hat{T}(\xi) = \hat{T}^{(0)}(\xi) / \hat{T}^{(1)}(\xi), \quad \hat{T}^{(\alpha)}(\xi) = T^{(\alpha)}(e^{i\xi}), \quad \alpha = 0, 1.$$

By (2.8) we have $\hat{T}^{(1)}(\xi) \neq 0$, thus $\hat{T}(\xi)$ is well defined. Since the boundary-scheme is stable then $|\hat{T}(\xi)| \leq 1$, so finally

$$(4.6) \quad |\hat{T}^{(0)}(\xi)| \leq |\hat{T}^{(1)}(\xi)| \neq 0 \quad \forall \xi.$$

The boundary-function associated with (2.6) satisfies

$$b(z, \kappa) = \sum_{j=0}^s (z c_j^{(1)} - c_j^{(0)}) \kappa^j = z T^{(1)}(\kappa) - T^{(0)}(\kappa).$$

So, for $|z| > 1$, we use (4.6) to find that

$$\begin{aligned} |b(z, e^{i\xi})| &= |z T^{(1)}(e^{i\xi}) - T^{(0)}(e^{i\xi})| = |z \hat{T}^{(1)}(\xi) - \hat{T}^{(0)}(\xi)| \\ &\geq |z| \cdot |\hat{T}^{(1)}(\xi)| - |\hat{T}^{(0)}(\xi)| > 0. \end{aligned}$$

That is, the equation

$$(4.7) \quad b(z, \kappa) = 0 \quad \text{with} \quad |z| > 1,$$

has no roots κ with $|\kappa| = 1$. Since the roots of (4.7) are continuous functions of z , we conclude that the number of κ with $|\kappa| < 1$ is fixed for $|z| > 1$, and can be determined by considering large values of $|z|$. Writing (4.7) in the form

$$T^{(1)}(\kappa) - z^{-1}T^{(0)}(\kappa) = 0,$$

we let $|z| \rightarrow \infty$ and use (2.8) to find that (4.7) has no roots in the unit disc. In other words, if $|z| > 1$ and $b(z, \kappa) = 0$, then κ must satisfy $|\kappa| > 1$. By continuity therefore, if $|z| \geq 1$ and $b(z, \kappa) = 0$, then $|\kappa| \geq 1$; i.e.,

$$b(z, \kappa) \neq 0, \quad \forall |z| \geq 1, \quad |\kappa| < 1.$$

This implies (4.2), and Theorem 4.1 completes the proof.

REFERENCES

1. M. Goldberg, On a boundary extrapolation theorem by Kreiss, Math. Comp., Vol. 31 (1977), 469-477.
2. B. Gustafsson, H. -O. Kreiss and A. Sundström, Stability theory of difference approximations for mixed initial boundary value problems.II, Math. Comp., Vol. 26 (1972), 649-686.
3. H. -O. Kreiss, Difference approximations for hyperbolic differential equations, Numerical Solution of Partial Differential Equations, (Proc. Sympos., Univ. of Maryland, College Park, Maryland 1965). Academic Press, New York, 1966, pp. 51-58.
4. H. -O. Kreiss, Stability theory for difference approximations of mixed initial boundary value problems. I, Math. Comp., Vol. 22 (1968), 703-714. MR 39 #2355.
5. G. Skölleremo, How the boundary conditions affect the stability and accuracy of some implicit methods for hyperbolic equations, Report #62, 1975, Dept. of Computer Science, Uppsala University.

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20. Abstract

(cont.) The results imply that certain well known boundary conditions, when used in combination with any (stable) dissipative scheme, always maintain stability.

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